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# Lévy flights with quenched noise amplitudes

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**Abstract.** We study the one-dimensional random walk of a particle in the presence of a short-range correlated quenched random field of jump lengths $l(x)$ drawn from a Lévy type distribution $p(l) \sim l^{-1-f}$ with $0 < f < 2$. We find the stochastic dynamics to be characterized by a novel length–time scaling relation that is caused by an effective jump-length distribution $p_{\rm eff}(l) \sim l^{-1-g}$ in the stationary state, which decays more rapidly than $p(l)$, i.e. $g \geqslant f$. For $f \gtrsim 1.3$, $g$ becomes larger than 2 and the particle diffuses normally although $p(l)$ has no finite second moment. A scaling theory is developed that describes the dynamical crossover from the annealed to the quenched situation.

Lévy flights [1] constitute a non-Brownian random motion, which can be most easily visualized by considering a particle that at regularly spaced time intervals performs jumps in random directions with jump lengths $l$ drawn from a distribution exhibiting a slowly decaying algebraic tail, $p(l) \sim l^{-1-f}$ where $0 < f < 2$. Since the second moment of such a distribution is infinite, there exists no diffusive lengthscale in the process and a time-dependent mean-square displacement is not well defined†. Nevertheless, a length–time scaling relation can also be introduced in the Lévy flight problem by considering, for example, the mean time $t_p(L)$ the particle needs to first pass a given distance $L$. If the jump lengths are uncorrelated in time (annealed disorder) one finds $t_p(L) \sim L^f$ [3] and this scaling also manifests itself when considering other quantities characterizing the annealed Lévy flight process, for example $x/t^{1/f}$ is the scaling variable entering the Lévy stable laws describing the probability of finding the particle near a position $x$ at time $t$ [4].

Stochastic processes of Lévy type have found many applications [1], not only in physics (e.g. in fluid flows [5], miscelle dynamics [6], or self-organized criticality [7]) but also in biology [8, 9] or finance [10, 11]. Further insight regarding applications to real systems have been gained by considering (annealed) Lévy flights with a truncated jump-length distribution [12] and Lévy flights in the presence of an additional quenched random force field [13]. Recently, the first calculations going beyond one-particle properties have been performed by studying the territory covered by many Levy flights [14].

In this paper we will investigate the question whether the fundamental length–time scaling valid in the annealed case also holds true when the random jump lengths are not drawn freshly at each timestep but are fixed in space (quenched disorder). As a starting point we will consider here a one-dimensional situation, i.e. the random flight of a particle

† This drawback can be resolved by assuming that larger jumps need longer times as it was done in the theory of Lévy walks, see [2].

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on a line with coordinate $x$ in the presence of a random field of jump lengths $l(x)$. The particle jumps with equal probability to the left or right at discrete timesteps $\Delta t = 1$ with a jump length $l(x_t)$ given by its current position $x_t$ at time $t$. Accordingly, the random motion of the particle can be described by

$$x_{t+1} = x_t + l(x_t)\eta_t \tag{1}$$

where $\eta_t = \pm 1$ is a random number specifying the jump direction (in the annealed case one would have $x_{t+1} = x_t + l_t\eta_t$ instead, where $l_t$ is independent of $x_t$). Because the particle will almost never visit *exactly* the same point in space again, we require that the random field $l(x)$ is correlated over some distance $a$ for the stochastic dynamics to be different from the annealed case. To this end we assign a random jump length $l_n$ to each space interval $na \leqslant x < (n+1)a$, $n = 0, \pm 1, \pm 2, \ldots$, where $l_n$ can assume values between a shortest jump length $l_{\min}$ and infinity, $l_{\min} \leqslant l_n < \infty$. Each interval can be considered as a 'site' on a linear chain with lattice constant $a$. For simplicity we focus on the situation where $l_{\min}$ is of the order of $a$ and accordingly set $l_{\min} = a = 1$. The jump lengths are then drawn from a distribution given explicitely by

$$p(l) = f l^{-1-f} \qquad 1 \leqslant l < \infty. \tag{2}$$

To explore the length–time scaling in the quenched case we perform computer simulations of the stochastic dynamics given by equation (1). First we study the mean first passage time $t_p(L)$ averaged over both several random trajectories for a given random field $l(x)$ and many different realizations of $l(x)$. In each trial the first passage time was obtained by putting the particle at the origin $x = 0$ at time $t = 0$ and by measuring the elapsed time, when it first passed one of the positions at $x = \pm L$. Figure 1($a$) shows

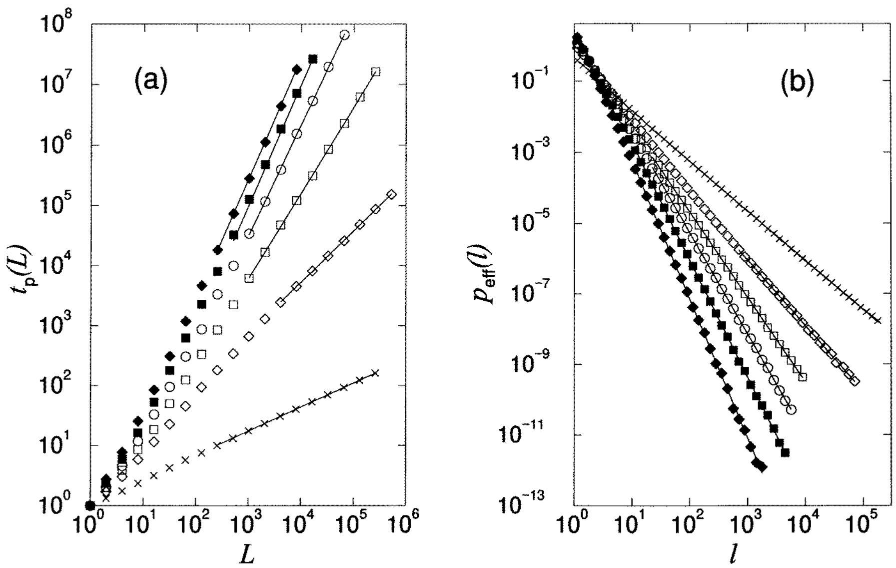


**Figure 1.** ($a$) Mean first passage time $t_p(L)$ as a function of $L$ and ($b$) effective jump-length distribution $p_{\text{eff}}(l)$ as a function of $l$ for $f = 0.4$ (×), 0.8 (◇), 1.0 (□), 1.2 (○), 1.5 (■), and 2.0 (◆). The full lines mark the asymptotic slopes $\alpha$ and $-1 - g$ in the double-logarithmic plots: $\alpha \cong 0.40$ ($g \cong 0.40$), 0.85 (0.86), 1.41 (1.41), 1.84 (1.86), 2.0 (2.30), and 2.0 (2.92) for $f = 0.4$, 0.8, 1.0, 1.2, 1.5, and 2.0, respectively. Averages were performed typically over $10^4$ different configurations.

$t_p(L)$ as a function of $L$ for various exponents $f = 0.4$, 0.8, 1, 1.5 and 2 (in a double-logarithmic representation). The straight lines in the double-logarithmic plot indicate a power-law behaviour for large $L$,

$$t_p(L) \sim L^{\alpha} \tag{3}$$

where the exponent $\alpha \cong f$ for $f = 0.4$ is in accordance with the result valid for the annealed case. For $f = 0.8$, 1 and 1.5, however, $\alpha$ is larger than $f$, which means that the superdiffusion is slowed down in comparison with the annealed case. Surprisingly, we obtain $\alpha \cong 2$ for $f = 1.5$, i.e. the particle diffuses normally although the prescribed jump-length distribution has no finite second moment. As a consequence, the curve for $f = 1.5$ in figure 1($a$) becomes, for large $L$, parallel to that for $f = 2.0$, where one would expect a normal diffusive behaviour (except for a possible logarithmic correction term).

The origin of the differences between $\alpha$ and $f$ can be understood by considering the *effective* jump-length distribution $p_{\text{eff}}(l)$ the particle encounters during its jump motion. This distribution was simply measured by counting the total number of jump lengths falling into a small interval during a time period that the particle needed to pass the largest distance $L$ shown in figure 1($a$). The result is shown in figure 1($b$) for the same values of the exponent $f$ as in figure 1($a$). As can be seen from the figure, $p_{\text{eff}}(l)$ scales differently as $p(l)$,

$$p_{\text{eff}}(l) \sim l^{-1-g} \tag{4}$$

with an exponent $g \geqslant f$ (we note that this is strictly true only in the infinite time limit, as will be discussed further below). Interestingly, the exponent $\alpha$ is equal to $g$ for $g \leqslant 2$ (within the numerical uncertainties), while we obtain $\alpha \cong 2$ for $g > 2$, where the effective distribution has a finite second moment (at $f = 1.5$ and $f = 2.0$). An analogous behaviour was also obtained for other values of $f$, i.e. the relation between $\alpha$ and $g$ in the quenched case is the same as the relation between $\alpha$ and $f$ in the annealed case,

$$\alpha = \begin{cases} g & \text{for } g \leqslant 2 \\ 2 & \text{for } g > 2. \end{cases} \tag{5}$$

We conclude that to understand the length–time relation in the quenched case, one has to explain why $p_{\text{eff}}(l)$ differs from $p(l)$.

Figure 2 shows the relation between the exponents $g$ and $f$. In the most interesting range $0 < f < 2$ we can distinguish between three different regimes: (i) for $f \lesssim 0.7$ we find $g \cong f$, i.e. the same superdiffusive behaviour as in the annealed case, (ii) for $0.7 \lesssim f \lesssim 1.3$ we obtain $f < g < 2$, i.e. a superdiffusive behaviour that is slowed down in comparison with the annealed case, and (iii) for $1.3 \lesssim f < 2$ we obtain $g > 2 > f$, i.e. a normal diffusive behaviour. From the inset of the figure, which shows $g$ over an extended range of $f$ values, we see that $g \to 1 + f$ when $f$ becomes much larger than 2.

The difference between $p_{\text{eff}}(l)$ and $p(l)$ is at first sight surprising, since at time $t = 1$ the particle initially placed at $x = 0$ has performed one jump only, with a jump length drawn from the prescribed distribution, and accordingly both $p_{\text{eff}}(l)$ and $p(l)$ must be identical. Thus $p_{\text{eff}}(l)$ has to change with time and we should consider a time-dependent distribution $p_{\text{eff}}(l, t)$. The question arises if $g$ (and $\alpha$) in figure 1 characterize the true stationary behaviour, i.e. if $p_{\text{eff}}(l, t \to \infty) \sim l^{-1-g}$.

To answer this question we consider the cumulative distribution $\pi_{\text{eff}}(l, t) = \int_l^{\infty} p_{\text{eff}}(l', t)\, \text{d}l'$, which is shown in figure 3($a$) as a function of $l$ for $f = 1.1$ and five different times $t$. (The following analysis may be done in full analogy with $p_{\text{eff}}(l, t)$, but with $\pi_{\text{eff}}(l, t)$ it can be more simply formulated.) From the figure it can be seen that for $l$ smaller than a time-dependent crossover length $l_x(t)$ the curves exhibit a power-law decay in

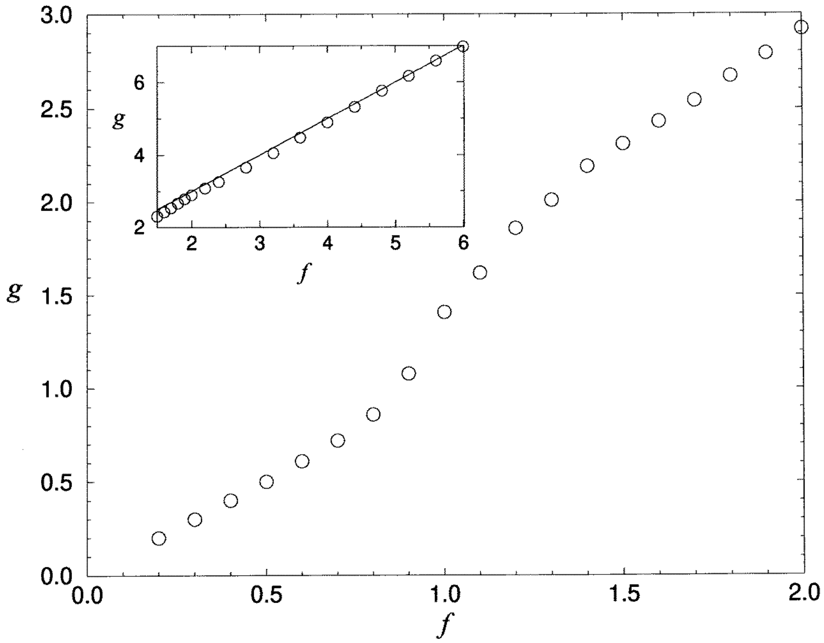


**Figure 2.** The exponent $g$ characterizing the effective jump-length distribution $p_{\mathrm{eff}}(l)$ (equation (4)) as a function of the exponent $f$ that defines the prescribed distribution $p(l)$ (equation (2)). The inset shows the behaviour for large $f$ values between 1.5 and 6 and the full line indicates the expected asymptotic behaviour $g = 1 + f$ for $f \gg 2$ (see text).

agreement with the exponent $g$ found in figure 1($a$), $\pi_{\mathrm{eff}}(l, t) \sim l^{-g}$, while for $l \gg l_x(t)$ they decay corresponding to the prescribed distribution, $\pi_{\mathrm{eff}}(l, t) \sim \pi(l) = \int_l^\infty p(l')\,\mathrm{d}l' = l^{-f}$. With increasing time $l_x(t)$ increases, and hence $g$ indeed corresponds to the stationary case†.

Assuming that $l_x(t) \sim t^\gamma$ we can make the following scaling ansatz,

$$\pi_{\mathrm{eff}}(l, t) = \frac{f(lt^{-\gamma})}{f(t^{-\gamma})} \tag{6}$$

where $f(y) \sim y^{-g}$ for $y \ll 1$ and $f(y) \sim y^{-f}$ for $y \gg 1$, and the denominator $f(t^{-\gamma})$ is required by the normalization condition $\pi_{\mathrm{eff}}(1, t) = 1$. To determine the exponent $\gamma$, let us consider the time dependence of $\pi_{\mathrm{eff}}(l, t)$ for a fixed $l_0 \gg l_x(t)$. Choosing $l_0$ large enough, we can assume that jump lengths larger than $l_0$ have no chance of being visited more than once in all configurations. Then the mean number $N(l_0, t)$ of sites with $l \geqslant l_0$ visited by the particle up to time $t$ must be given by $N(l_0, t) \simeq \pi(l_0)S(t)$, where $S(t)$ is the mean number of *distinct* visited sites up to time $t$. Hence, for $l \gtrsim l_0 \gg l_x(t)$ we obtain $\pi_{\mathrm{eff}}(l, t) = N(l_0, t)/t \simeq \pi(l)S(t)/t \sim l^{-f}t^{1/\nu-1}$, where $\nu \geqslant 1$ is the exponent characterizing the asymptotic time dependence of $S(t) \sim t^{1/\nu}$. On the other hand, we obtain from equation (6) $\pi_{\mathrm{eff}}(l, t) \sim l^{-f}t^{-\gamma(g-f)}$ for $l \gg t^\gamma \gg 1$, and hence by comparison

$$\gamma = \frac{\nu - 1}{(g - f)\nu}. \tag{7}$$

† We note that we also performed simulations for a periodic line, where one can reach the stationary state before the measurements of the relevant quantities are done. As expected, when starting from the stationary state, we found the same asymptotic scaling behaviour as in the case where initially the particle is put at the origin.

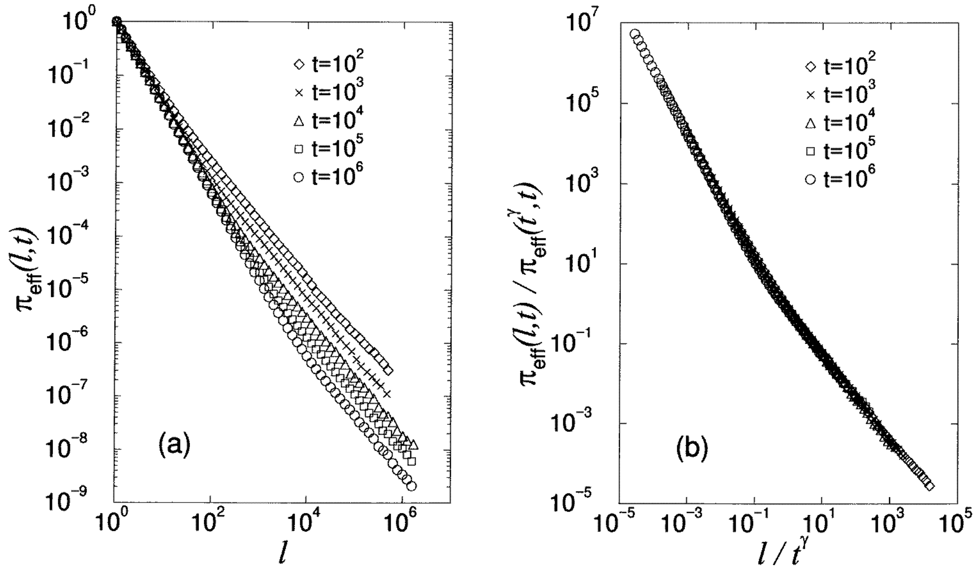


**Figure 3.** (*a*) The cumulative effective probability distribution $\pi_{\text{eff}}(l, t)$ as a function of $l$ for $f = 1.1$ and five different times $t$. For small $l \ll l_x(t)$, $\pi_{\text{eff}}(l, t) \sim l^{-g}$ with $g \cong 1.62$, while for $l \gg l_x(t)$, $\pi_{\text{eff}}(l, t) \sim l^{-f}$. (*b*) Scaling plot of the data shown in (*a*) with $\gamma = (\nu - 1)/[(g - f)\nu] = (g - 1)/[(g - f)g] \cong 0.74$ (see equations (7), (8)).

To derive the exponent $\nu$ let us ask when the typical length $L(t) \sim t^{1/\alpha}$ passed by the particle during the time $t$ should scale as the mean number of distinct visited sites $S(t)$. Clearly, this can be true only as far as the particle has no chance during the time $t$ to encounter a jump length of order $L$. The typical *largest* jump length encountered by the particle within the time $t$ is $l_{\max}(t) \sim S(t)^{1/f} \sim t^{1/\nu f}$. Hence for $l_{\max}(t) < L(t)$, i.e. $f > \alpha/\nu$ we expect $\nu = \alpha$, i.e. for $f > 1$. For $f \leqslant 1$, however, the length $L(t)$ is actually passed due to the fact that the particle encounters a jump length of order $L$, i.e. it must hold $l_{\max}(t) \sim L(t)$, from which follows $\nu = \alpha/f$. In summary we thus obtain

$$\nu = \begin{cases} \alpha/f = g/f & \text{for } f \leqslant 1 \\ \alpha & f > 1. \end{cases} \tag{8}$$

Note that we confirmed the validity of equation (8) in the diffusive and superdiffusive regime by numerical calculations.

To test the scaling ansatz for the data shown in figure 3(*a*) ($f = 1.1$) we hence choose $\nu = \alpha = g$ to evaluate $\gamma$, and plot in figure 3(*b*) $\pi_{\text{eff}}(l, t)/\pi_{\text{eff}}(t^\gamma, t) = f(lt^{-\gamma})/f(1)$ as a function of the scaling variable $lt^{-\gamma}$. As can be seen from the figure, a very good data collapse is obtained. We tested the scaling ansatz also for other exponents $f$ and always found as good agreement as in figure 3(*b*).

Let us now discuss the question why $g$ can be different from $f$. Unfortunately we are not able to give an explicit analytical formula for the relation between $g$ and $f$ shown in figure 2. However, insight into the mechanism leading to the different values of the two exponents can be gained by considering an analogous lattice model, where the jump lengths are restricted to integer numbers (i.e. $p(l) \propto l^{-1-f}$ with $l = 1, 2, 3, \ldots$). To visualize the random flights on such a chain, it is convenient to distinguish between two types of sites for a given jump length $l \geqslant 2$: $A_l$ sites with jump length smaller than $l$ and $B_l$ sites having jump length larger than or equal to $l$. Clearly, the concentration $\pi(l) = \sum_{l'=l}^{\infty} p(l')$ of $B_l$ sites scales as $\pi(l) \sim l^{-f}$, i.e. it becomes very small for large $l$. Hence, for any given $f$ one can find a sufficiently large $l$, where the $B_l$ sites can be viewed as isolated sites separating $A_l$-clusters of neighbouring $A_l$ sites. The mean size $\sigma_l$ of the $A_l$ clusters scales as the mean distance between two $B_l$ sites, i.e. $\sigma_l \sim 1/\pi(l) \sim l^f$.

When $f$ is large ($f \gg 2$), almost all sites of the $A_l$-clusters have jump length 1 and almost all of the $B_l$ sites have jump length $l$. Hence, for a given $l$ we can view the random motion of the particle as taking place within $A_l$-clusters of size $\sigma_l$ with jump length 1, interrupted by visits to $B_l$ sites with jump length $l$ that mark the boundaries of the $A_l$-clusters. Following [15] we can write $\pi_{\text{eff}}(l) \sim 1/n_1(\sigma_l, l)$ in such a simplified situation, where $n_1(\sigma_l, l)$ is the average number of jumps the particle needs to escape a cluster of size $\sigma_l$ consisting of sites with jump length 1, after it entered this cluster at a position a distance $l$ apart from its boundary. This number $n_1(\sigma_l, l)$ can be calculated analytically [15] and it scales as $n_1(\sigma_l, l) \sim \sigma_l l \sim l^{1+f}$. Accordingly, $\pi_{\text{eff}}(l) \sim l^{-1-f}$, i.e. $g = 1 + f$ in agreement with the numerical results presented in figure 2. We see that the reason for the difference between $g$ and $f$ is the fact that the particle motion is slowed down in spatial fluctuations with small jump lengths such that these small jump lengths are more often encountered by the particle than expected from the prescribed distribution.

The problem becomes more complicated when $f$ gets smaller. Already for $f \lesssim 4$ the $A_l$-clusters can no longer be considered as consisting of sites with jump length 1 only. Then the derivation above is no longer applicable because one needs to take into account that the $B_l$ sites marking the boundaries of the $A_l$-clusters can be overjumped. For $f \ll 1$ finally, the mean distance $\sigma_l \sim l^f$ between two $B_l$ sites becomes much smaller than the typical jump length $l$ of a $B_l$ site and the picture of a transport behaviour governed by a random motion among neighbouring $A_l$-clusters loses its meaning. The particle explores new space regions within a time period proportional to $t$ (i.e. $S(t) \sim t$) and the stochastic dynamics becomes essentially the same as in the annealed case.

In summary we have shown that Lévy flights with quenched jump-length distributions $p(l) \sim l^{-1-f}$ exhibit a length–time scaling relation, which in general is different from the annealed situation. The quenched case differs from the annealed case because the particle stays on average longer in spatial regions with small jump lengths than in those with larger ones, which causes the effective jump-length distribution $p_{\text{eff}}(l)$ to decay faster than $p(l)$, i.e. $p_{\text{eff}}(l) \sim l^{-1-g}$ with $g \geqslant f$. One might ask if different scaling properties of the quenched and annealed situation might still occur for Lévy flights in higher dimensions $d$, where one would expect the fluctuation effects to become less important. In other words, there should, for a given $f$ (in the interesting regime $f < 2$), exist a critical dimension $d_c$, above which the diffusion in the presence of queched noise amplitudes is no longer slowed down in comparison with the annealed case.

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## References

[1] Shlesinger M F, Zaslavsky G M and Frisch U (ed) 1995 *Lévy Flights and Related Topics in Physics (Lecture*

*Notes in Physics 450)* (Heidelberg: Springer) p 196
Klafter J, Shlesinger M F and Zumofen G 1996 *Phys. Today* **49** 33
[2] Shlesinger M F, West B J and Klafter J 1987 *Phys. Rev. Lett.* **58** 1100
[3] This result can be readily derived form the treatment of Gillis J E and Weiss G H 1970 *J. Math. Phys.* **11** 1307
[4] Bouchaud J P and Georges A 1990 *Phys. Rep.* **195** 128
[5] Solomon T H, Weeks E R and Swinney H L 1993 *Phys. Rev. Lett.* **71** 3975
Weeks E R, Urbach J S and Swinney H L 1996 *Physica* **97D** 291
Venkataramani S C, Antonsen T M Jr and Ott E 1997 *Phys. Rev. Lett.* 1997 **78** 3864
[6] Ott A, Bouchaud J P, Langevin D and Urbach W 1990 *Phys. Rev. Lett.* **65** 2201
[7] Christensen K, Corral Á, Frette V, Feder J and Jøssang T 1996 *Phys. Rev. Lett.* **77** 107
Boguñá M and Corral Á 1997 *Phys. Rev. Lett.* **78** 4950
[8] Kopf M, Corinth C, Haferkamp O and Nonnenmacher T F 1996 *Biophys. J.* **70** 2950
[9] Viswanathan G M, Afanasyev V, Buldyrev S V, Murphy E J, Prince P A and Stanley H E 1996 *Nature* **381** 413
[10] Mantegna R N and Stanley H E 1996 *Nature* **383** 587
1995 Mantegna R N and Stanley H E *Nature* **376** 46
[11] Bouchaud J-P and Sornette D 1994 *J. Physique* I **4** 863
[12] Mantegna R N and Stanley H E 1994 *Phys. Rev. Lett.* **73** 2946
[13] Fogedby H C 1994 *Phys. Rev. Lett.* **73** 2517
[14] Berkolaiko G, Havlin S, Larralde H and Weiss G H 1996 *Phys. Rev.* E **53** 5774
Berkolaiko G and Havlin S 1997 *Phys. Rev.* E **55** 1395
[15] Kutner R and Maass P 1997 *Phys. Rev.* E **55** 71